GROWTH SEQUENCES FOR CIRCLE DIFFEOMORPHISMS

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Abstract. We obtain results on the growth sequences of the differential for iterations of circle diffeomorphisms without periodic points.

1. Introduction and statement of results

Let $f : S^1 \to S^1$ be a C^1-diffeomorphism where $S^1 = \mathbb{R}/\mathbb{Z}$. We define the *growth sequence* for f by

$$\Gamma_n(f) = \max\{\|Df^n\|, \|Df^{-n}\|\}, \quad n \in \mathbb{N},$$

where f^n is the n-th iteration of f and $\|Df^n\| = \max_{x \in S^1} |Df^n(x)|$.

If f has periodic points, then the study of growth sequences reduces to the case of interval diffeomorphisms which was studied in [B],[PS],[W].

If f has no periodic points, then by the theorem of Gottschalk-Hedlund $\Gamma_n(f)$ is bounded if and only if f is C^1-conjugate to a rotation. Notice that if $\Gamma_n(f)$ is bounded then f is minimal. So it is natural to ask how rapidly could the sequence $\Gamma_n(f)$ grow if it is unbounded.

In this paper we give an answer to this question:

Theorem 1. *Let $f : S^1 \to S^1$ be a C^2-diffeomorphism without periodic points. Then*

$$\lim_{n \to \infty} \frac{\Gamma_n(f)}{n^2} = 0.$$

Theorem 2. *For any increasing unbounded sequence of positive real numbers $\theta_n = o(n^2)$ as $n \to \infty$ and any $\varepsilon > 0$ there exists an analytic diffeomorphism $f : S^1 \to S^1$ without periodic points such that*

$$1 - \varepsilon \le \limsup_{n \to \infty} \frac{\Gamma_n(f)}{\theta_n} \le 1.$$

2. Preliminaries

Given an orientation preserving homeomorphism $f : S^1 \to S^1$, its *rotation number* is defined by

$$\rho(f) = \lim_{n \to \infty} \frac{\tilde{f}^n(x) - x}{n} \mod \mathbb{Z}$$

where $\tilde{f}$ denotes a lift of f to $\mathbb{R}$. The limit exists and is independent on $x \in \mathbb{R}$ and a lift $\tilde{f}$.

Put $\alpha = \rho(f)$. Let R_α be the rigid rotation by α

$$R_\alpha(x) = x + \alpha \mod \mathbb{Z}.$$

For the basic properties of circle homeomorphisms and the combinatorics of orbits of the rotation of the circle, general references are [MS] chapter I and [KH] chapter 11, 12.

By Poincaré the order structure of orbits of f and R_α on S^1 are almost same. In particular if $\rho(f) = \frac{p}{q} \in \mathbb{Q}/\mathbb{Z}$ then f has periodic points of period q and every periodic orbits of f have the same order as orbits of $R_{\frac{p}{q}}$ on S^1. $\rho(f) \in (\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}$ if and only if f has no periodic points, in this case, if f is of class C^2 then by the well known theorem of Denjoy f is topologically conjugate to R_α.

Suppose $\alpha \in (\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}$. Let

$$\alpha = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \quad , a_i \geq 1, a_i \in \mathbb{N}$$

be the continued fraction expansion of α, and

$$\frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n]$$

be its n-th convergent. Then p_n and q_n satisfy

$$p_{n+1} = a_{n+1}p_n + p_{n-1},\ p_0 = 0,\ p_1 = 1,$$
$$q_{n+1} = a_{n+1}q_n + q_{n-1},\ q_0 = 1,\ q_1 = a_1,$$
$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \alpha < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

The sequence of rational numbers $\{\frac{p_n}{q_n}\}$ is the best rational approximation of α. This can be expressed using the dynamics of R_α as follows. $R_\alpha^{q_n}(0) \in [0, R_\alpha^{-q_{n-1}}(0)]$, and if $k > q_{n-1}$, $R_\alpha^k(0) \in [R_\alpha^{q_{n-1}}(0), R_\alpha^{-q_{n-1}}(0)]$ then $k \geq q_n$. Note that for $0 \leq k \leq a_{n+1}, R_\alpha^{kq_n}(0) \in [0, R_\alpha^{-q_{n-1}}(0)]$, and $R_\alpha^{(a_{n+1}+1)q_n}(0) \notin [0, R_\alpha^{-q_{n-1}}(0)]$.

For $\alpha \in (\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}$ the continued fraction expansion is unique. On the other hand for $\beta \in \mathbb{Q}/\mathbb{Z}$ expressions by continued fractions are not unique, $\beta = [b_1, b_2, \ldots, b_n + 1] = [b_1, b_2, \ldots, b_n, 1]$.

For $\alpha = [a_1, a_2, \ldots]$ and $i, j \in \mathbb{N}, 1 \leq i \leq j$ we denote $\alpha|[i,j] = [a_i, a_{i+1}, \ldots, a_j]$. In case we emphasize α we denote $a_i(\alpha), p_i(\alpha), q_i(\alpha)$.

For $x \in S^1$, $I_n(x)$ denotes the smaller interval with endpoints x and $f^{q_n}(x)$ and for an interval $J \subset S^1$, $|J|$ the length of J.

The following is well known. See [MS] chapter I section 2a.

Lemma 1. (Denjoy) *Let f be a C^1-diffeomorphism of S^1 without periodic points and $\log Df : S^1 \to \mathbb{R}$ has bounded variation. Then there exists a positive constant $C_1 = C_1(f)$ satisfying the following properties.*

(1) *For any $0 \leq l \leq q_{n+1}$ and for every $x_1, x_2 \in I_n(x)$*

$$\frac{1}{C_1} \leq \frac{Df^l(x_1)}{Df^l(x_2)} \leq C_1.$$

(2) (Denjoy inequality) *For every $n \in \mathbb{N}$,*

$$\frac{1}{C_1} \leq \|Df^{q_n}\| \leq C_1.$$

As stated in section 1, the growth sequences play a significant role in the problem of the smooth linearization of circle diffeomorphisms, where the arithmetic property of rotation numbers and the regularity of diffeomorphisms are important. This problem has a rich history, see e.g. [A], [H], [Y], [KS], [St], [KO].

In this paper, particularly we need the following improvement of Denjoy inequality which is due to Katznelson and Ornstein. The statement of Lemma 2 is obtained by merging results in [KO], for (1), (1.16), lemma 3.2 (3.6) and proposition 3.3 (a), for (2), theorem 3.7.

Lemma 2. *Let f be a C^2-diffeomorphism of S^1 without periodic points. Set*

$$E_n = \max\{\|\log Df^{q_n}\|, \max_{x\in S^1}\{|D\log Df^{q_n}(x)||I_{n-1}(x)|\}\}.$$

Then the following hold.

(1) $\lim_{n\to\infty} E_n = 0$.

(2) *If f is of class $C^{2+\delta}, \delta > 0$ then there exist $C > 0$ and $0 < \lambda < 1$ such that $\|\log Df^{q_n}\| \le C\lambda^n$ for any $n \in \mathbb{N}$.*

The conclusion of Lemma 2 (2) plus some arithmetic condition of $\rho(f)$ are sufficient to provide the C^1-linearization of f. We need the following which is a special case of the main theorem in [KO]. For $C^{3+\delta}$-diffeomorphisms it is originally due to Herman [H].

Corollary of Lemma 2 (2). *If f is of class $C^{2+\delta}$ and the rotation number $\alpha = \rho(f)$ is of bounded type i.e. $a_i(\alpha)$ is uniformly bounded then $\|Df^n\|$ is uniformly bounded.*

3. Proof of Theorem 1

Let $f : S^1 \to S^1$ be a C^2-diffeomorphism without periodic points with the rotation number $\rho(f) = [a_1, a_2, \ldots]$ and its convergents $\{\frac{p_n}{q_n}\}$.

The following crucial and fundamental lemma is due to Polterovich and Sodin ([PS] lemma 2.3).

Lemma 3. (Growth lemma) *Let $\{A(k)\}_{k\ge 0}$ be a sequence of real numbers such that for each $k \ge 1$*

$$2A(k) - A(k-1) - A(k+1) \le C\exp(-A(k)), \quad C > 0,$$

and $A(0) = 0$. Then either for each $k \ge 0$

$$A(k) \le 2\log\left(k\sqrt{\frac{C}{2}} + 1\right), \text{ or } \liminf_{k\to\infty}\frac{A(k)}{k} > 0.$$

Lemma 4. *For $0 \le k \le a_{n+1}+1$ we set $A_n(k) = \log\|Df^{kq_n}\|$. Then there exists a positive constant $C = C(f)$ independent with n such that for $1 \le k \le a_{n+1}$,*

$$2A_n(k) - A_n(k-1) - A_n(k+1) \le CE_n\exp(-A_n(k)).$$

Proof. Let $A_n(k) = \log Df^{kq_n}(x_0)$ and $x_i = f^{iq_n}(x_0)$. Then we have,

$$\begin{aligned}
&2A_n(k) - A_n(k-1) - A_n(k+1)\\
&\le 2\log Df^{kq_n}(x_0) - \log Df^{(k-1)q_n}(x_1) - \log Df^{(k+1)q_n}(x_{-1})\\
&\le |\log Df^{q_n}(x_0) - \log Df^{q_n}(x_{-1})| = |D\log Df^{q_n}(y_0)||I_n(x_{k-1})|\frac{|I_n(x_{-1})|}{|I_n(x_{k-1})|},
\end{aligned}$$

where $y_0 \in I_n(x_{-1})$.

Notice that the intervals $I_n(x_{-1}), I_n(x_0), I_n(x_1), \ldots, I_n(x_{a_{n+1}-1})$ are adjacent in this order and $\cup_{i=0}^{a_{n+1}-1} I_n(x_i) \subset I_{n-1}(f^{-q_{n-1}}(x_0))$. Since $y_0 \in I_n(x_{-1})$, we have for $1 \leq k \leq a_{n+1}-1$, $I_n(x_{k-1}) \subset I_{n-1}(f^{-q_{n-1}}(y_0))$. So by Denjoy inequality (Lemma 1 (2)) we have

$$|I_n(x_{k-1})| \leq C_1^2 |I_{n-1}(y_0)|,$$

and using lemma 1 (1) we have

$$\frac{|I_n(x_{-1})|}{|I_n(x_{k-1})|} \leq C_1 \frac{1}{Df^{kq_n}(x_0)}.$$

Hence we have

$$\begin{aligned} &2A_n(k) - A_n(k-1) - A_n(k+1) \\ &\leq C_1^3 |D \log Df^{q_n}(y_0)||I_{n-1}(y_0)| \frac{1}{Df^{kq_n}(x_0)} \leq C_1^3 E_n \exp(-A_n(k)). \end{aligned}$$

□

We extend $A_n(k)$ for $k \geq a_{n+1}+2$ by $A_n(k) = A_n(a_{n+1}+1)$. Then by Lemma 1 (2) and the definition of E_n we have

$$\begin{aligned} &2A_n(a_{n+1}+1) - A_n(a_{n+1}) - A_n(a_{n+1}+2) \\ &\leq \log Df^{(a_{n+1}+1)q_n}(x_0) - \log Df^{a_{n+1}q_n}(x_0) \leq \|\log Df^{q_n}\| \\ &\leq E_n \exp(-A_n(a_{n+1}+1)) \|Df^{(a_{n+1}+1)q_n}\| \\ &\leq E_n \exp(-A_n(a_{n+1}+1)) \|Df^{q_{n+1}}\| \|Df^{q_n}\| \|Df^{-q_{n-1}}\| \\ &\leq C_1^3 E_n \exp(-A_n(a_{n+1}+1)). \end{aligned}$$

For $k \geq a_{n+1}+2$, $2A_n(k) - A_n(k-1) - A_n(k+1) = 0$.

Then since $A_n(k)$ satisfy the condition of Lemma 3 with the constant $C = C_1^3$ and obviously $\lim_{k\to\infty} \frac{A_n(k)}{k} = 0$, we have

$$\|Df^{kq_n}\| \leq \left(\sqrt{\frac{CE_n}{2}} k + 1 \right)^2, \ 0 \leq k \leq a_{n+1}.$$

For $q_n \leq l < q_{n+1}$, we define $0 \leq k_{i+1} \leq a_{i+1}, (i = 0, 1, \ldots, n)$ inductively by

$$r_{n+1} = l, \ r_{i+1} = k_{i+1} q_i + r_i, \ 0 \leq r_i < q_i.$$

Then, using $\frac{q_{i+1}}{q_i} \geq a_{i+1} \geq k_{i+1}$,

$$\begin{aligned} \frac{\|Df^l\|}{l^2} &\leq \frac{\prod_{i=0}^n \|Df^{k_{i+1}q_i}\|}{(k_{n+1}q_n)^2} \leq \frac{\prod_{i=0}^n \left(\sqrt{\frac{CE_i}{2}} k_{i+1} + 1 \right)^2}{\left(k_{n+1} \prod_{i=0}^{n-1} \frac{q_{i+1}}{q_i} \right)^2} \\ &\leq \left(\sqrt{\frac{CE_n}{2}} + 1 \right)^2 \prod_{i=0}^{n-1} \left(\sqrt{\frac{CE_i}{2}} + \frac{q_i}{q_{i+1}} \right)^2. \end{aligned}$$

Since $\frac{q_i}{q_{i+2}} < \frac{1}{2}$, for sufficiently small E_i and E_{i+1}

$$\left(\sqrt{\frac{CE_i}{2}} + \frac{q_i}{q_{i+1}}\right)\left(\sqrt{\frac{CE_{i+1}}{2}} + \frac{q_{i+1}}{q_{i+2}}\right) \le \frac{1}{2}.$$

By Lemma 2 (1), $E_n \to 0$ as $n \to \infty$. Consequently we have

$$\lim_{l\to\infty} \frac{\|Df^l\|}{l^2} = 0.$$

For the case $\|Df^{-l}\|, l > 0$, the argument is the same.

4. Proof of Theorem 2

Let $\{\theta_n\}_{n\ge1}$ be any increasing unbounded sequence of positive real numbers such that $\theta_n = o(n^2)$ as $n \to \infty$.

We consider the two-parameter family of rational functions on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,

$$J_{a,t} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad J_{a,t}(z) = \exp(2\pi it)z^2 \frac{z+a}{az+1}$$

where $a \in \mathbb{R}, a > 3$ and $t \in \mathbb{R}/\mathbb{Z}$.

For each a, t the map $J_{a,t}$ makes invariant the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C}; |z| = 1\}, J_{a,t}(\partial\mathbb{D}) = \partial\mathbb{D}$, moreover the restriction of $J_{a,t}$ to $\partial\mathbb{D}$ is an orientation preserving diffeomorphism. The set of critical points of $J_{a,t}$ consists of four elements containing 0 and ∞ which are fixed by $J_{a,t}$. Notice that if $a \to \infty$ then on a compact tubular neighbourhood of the unit circle in $\mathbb{C} \setminus \{0\}$ $J_{a.t}$ uniformly converges to the rotation $z \mapsto \exp(2\pi it)z$.

Put $\psi : \mathbb{R}/\mathbb{Z} \to \partial\mathbb{D}, \psi(x) = \exp(2\pi ix)$. Conjugating $J_{a,t}|\partial\mathbb{D}$ by ψ we obtain the family of analytic circle diffeomorphisms $\{f_{a,t}\}$,

$$f_{a,t} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \ f_{a,t}(x) = \psi^{-1} \circ J_{a,t} \circ \psi(x) = f_{a,0}(x) + t \mod \mathbb{Z}.$$

Temporarily we fix $a > 3$ and abbreviate as $f_{a,t} = f_t$.

The following properties of this family are standard. See e.g. [MS] chapter I, section 4, where Arnold family $x \mapsto x + a\sin(2\pi x) + t$ is mainly dealt with but the argument is valid for our family. Also see [KH] chapter 11, section 1.

The map $F : S^1 \to S^1, t \mapsto \rho(f_t)$ is continuous and monotone increasing. We set

$$K = \{t \in S^1; \rho(f_t) \text{ is irrational}\}.$$

We denote $\mathrm{Cl}(K)$ the closure of K. $F|K$ is a one-to-one map. For $t \in K$ with $F(t) = \alpha$, we denote $f_t = \hat{f}_\alpha$. Notice that f_t never conjugate to a rational rotation. Hence for $\frac{p}{q} \in \mathbb{Q}/\mathbb{Z}$, $F^{-1}(\frac{p}{q})$ is a closed interval, say, $[\frac{p}{q}_-, \frac{p}{q}_+]$.

Moreover, $F^{-1}|(\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z} : (\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z} \to K$ is continuous and

$$\lim_{\alpha\to\frac{p}{q}-0} F^{-1}|(\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}(\alpha) = \frac{p}{q}_-, \ \lim_{\alpha\to\frac{p}{q}+0} F^{-1}|(\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}(\alpha) = \frac{p}{q}_+.$$

Note that for every $\frac{p}{q} \in \mathbb{Q}/\mathbb{Z}$ and every $x \in S^1$, there exists $t \in [\frac{p}{q}_-, \frac{p}{q}_+]$ such that $f_t^q(x) = x$. For $\frac{p}{q} \in \mathbb{Q}/\mathbb{Z}$, put $t_* = \frac{p}{q}_-$. The case $t_* = \frac{p}{q}_+$ is similar. Then the graph

of $f_{t_*}^q(x)$ touches from below to the graph of the identity map, in particular, there exists $x_0 \in S^1$ such that

$$f_{t_*}^q(x_0) = x_0,\ Df_{t_*}^q(x_0) = 1.$$

Then the following holds.

Lemma 5. $D^2 f_{t_*}^q(x_0) \neq 0$.

Proof. By contradiction, we suppose $D^2 f_{t_*}^q(x_0) = 0$. Then in our case $D^3 f_{t_*}^q(x_0) = 0$, otherwise x_0 is a topologically transversal fixed point of $f_{t_*}^q$ and persists under perturbation of f_{t_*}, which contradicts $t_* \in \mathrm{Cl}(K)\backslash K$. Set $z_0 = \psi(x_0) \in \partial\mathbb{D}$. Since the order of tangency to the identity map is an invariant of C^∞-conjugacy [T], we have for $J_{t_*} = J_{a,t_*}$

$$J_{t_*}^q(z_0) = z_0,\ DJ_{t_*}^q(z_0) = 1,\ D^2 J_{t_*}^q(z_0) = D^3 J_{t_*}^q(z_0) = 0.$$

So z_0 is a parabolic fixed point for $J_{t_*}^q$ with multiplicity at least four. See [M] chapter 7. By the Laeu-Fatou flower theorem ([M] th.7.2) z_0 has at least three basins of attraction for $J_{t_*}^q$. Let B be one of the immediate attracting basins of z_0 for $J_{t_*}^q$. Then B must contain at least one critical point of $J_{t_*}^q$ ([M] corollary 7.10). So each basin of the cycle $\{z_0, J_{t_*}(z_0), \ldots, J_{t_*}^{q-1}(z_0)\}$ contains at least one critical point of J_{t_*}. But J_{t_*} has exactly four critical points and two of them are fixed points. We obtain a contradiction. $\square$

Hence, for example, by comparing a fractional linear transformation (see also [B] thorem 1 (A)), we can see that there exist $C > 0$ and $\{x_l\}_{l\geq 1} \subset S^1$ with $\lim_{l\to\infty} x_l = x_0$ such that

$$Df_{t_*}^{lq}(x_l) \geq Cl^2,\ \text{for any } l \in \mathbb{N}.$$

Since $\theta_n = o(n^2)$, we have

Corollary of Lemma 5. *For sufficiently large l, we have $\|Df_{t_*}^{lq}\| > \theta_{lq}$.*

Remark. For each $k \in \mathbb{N}$ we set

$$U_k = \{t \in \mathrm{Cl}(K); \text{There exist } m \geq k \text{ and } x \in S^1 \text{ such that } Df_t^m(x) > m\sqrt{\theta_m}\}.$$

Obviously U_k is open set in $\mathrm{Cl}(K)$. By the corollary and the denseness of preimages of rational numbers by F in $\mathrm{Cl}(K)$, U_k is dense in $\mathrm{Cl}(K)$. So the following set is a residual subset of $\mathrm{Cl}(K)$,

$$\{t \in \mathrm{Cl}(K); \limsup_{n\to\infty} \frac{\Gamma_n(f_t)}{\theta_n} = \infty\}.$$

We seek a desired diffeomorphism in this family $\{f_t\}$ by specifying its rotation number $\alpha_\infty = \rho(f_{t_\infty}) \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$. We will define an increasing sequence of even numbers $0 < n_1 < n_2 < n_3 < \cdots$, and a sequence of positive integers $A_1, A_2, A_3, \ldots$ inductively. The continued fraction expansion of α_∞ is the following.

$$\alpha_\infty = [a_1(\alpha_\infty), a_2(\alpha_\infty), a_3(\alpha_\infty), \ldots]$$
$$= [1, 1, \ldots, 1, A_1, 1, \ldots, 1, A_2, 1, \ldots, 1, A_k, 1, \ldots]$$

where if $i = n_k$ then $a_i(\alpha_\infty) = A_k$ and if $i \neq n_k$ for any k then $a_i(\alpha_\infty) = 1$.

For $m, A \geq 1, m, A \in \mathbb{N}$, we set

$$\alpha_m^A = [a_1(\alpha_m^A), a_2(\alpha_m^A), a_3(\alpha_m^A), \ldots]$$

$$= [1, 1, \ldots, 1, A_1, 1, \ldots, 1, A_{m-1}, 1, \ldots, 1, A, 1, 1, 1, \ldots]$$

where $a_i(\alpha_m^A) = A_k$ if $i = n_k \le n_{m-1}$ and $a_i(\alpha_m^A) = A$ if $i = n_m$ and $a_i(\alpha_m^A) = 1$ otherwise.

Set $\alpha_m = \alpha_m^{A_m}$. Notice that $\alpha_m^A|[1, n_m - 1] = \alpha_\infty|[1, n_m - 1]$ and α_m^A is of bounded type. Unless otherwise stated we use the symbols p_n, q_n as $p_n(\alpha_\infty), q_n(\alpha_\infty)$.

Lemma 6. *There exist a sequence of even numbers $0 < n_1 < n_2 < n_3 < \cdots$, and a sequence of positive integers $A_1, A_2, A_3, \ldots$ such that for each $m \ge 1$ the following properties hold.*

(1) *For any $j \in \mathbb{Z}$ with $q_{n_m-1} \le |j| \le A_m q_{n_m-1}$, $\|D\hat{f}^j_{\alpha_m}\| < \theta_{|j|}$.*

(2) *There exists $j_m \in \mathbb{Z}$ such that*

$$q_{n_m-1} \le |j_m| \le (A_m + 1) q_{n_m-1}, \ \|D\hat{f}^{j_m}_{\alpha_m^{A_m+1}}\| \ge \theta_{|j_m|}.$$

(3) *For any $t \in F^{-1}(\alpha)$ with $\alpha|[1, n_{m+1} - 1] = \alpha_m|[1, n_{m+1} - 1]$ and any $j \in \mathbb{Z}$ with $|j| \le q_{n_m}$,*

$$\|Df_t^j\| - 1 \le \|D\hat{f}^j_{\alpha_m}\| \le \|Df_t^j\| + 1.$$

Proof. Let $\alpha_0 = [1, 1, 1, \ldots] = \frac{\sqrt{5}-1}{2}$. Since α_0 is of bounded type by Corollary of Lemma 2 (2) there exists $C_0 > 0$ such that for any $l \in \mathbb{Z}$, $\|D\hat{f}^l_{\alpha_0}\| \le C_0$. Let n_1 be a sufficiently large even number such that if $|i| \ge q_{n_1-1}(\alpha_0)$ then $\theta_{|i|} \ge C_0$.

Let $\beta_1 = \alpha_0|[1, n_1 - 1] = \frac{p_{n_1-1}(\alpha_0)}{q_{n_1-1}(\alpha_0)} = [1, 1, \ldots 1] = [1, 1, \ldots 1, \infty] \in \mathbb{Q}/\mathbb{Z}$. Then by Corollary of Lemma 5 there exists $d \in \mathbb{N}$ such that $\|Df^{dq_{n_1-1}}_{\beta_{1-}}\| > \theta_{dq_{n_1-1}}$, where $F^{-1}(\beta_1) = [\beta_{1-}, \beta_{1+}]$. Since $\alpha_1^A \to \beta_1 - 0$ as $A \to \infty$, $F^{-1}(\alpha_1^A) \to \beta_{1-}$ as $A \to \infty$. So for sufficiently large A we have $\|D\hat{f}^{dq_{n_1-1}}_{\alpha_1^A}\| > \theta_{dq_{n_1-1}}$. Hence the following is well defined.

$$A_1 = \max\{A; \text{for any } j \in \mathbb{Z} \text{ with } q_{n_1-1} \le |j| \le A q_{n_1-1}, \ \|D\hat{f}^j_{\alpha_1^A}\| < \theta_{|j|}\}.$$

Therefore there exists $j_1 \in \mathbb{Z}$ such that

$$q_{n_1-1} \le |j_1| \le (A_1 + 1) q_{n_1-1}, \ \|D\hat{f}^{j_1}_{\alpha_1^{A_1+1}}\| \ge \theta_{|j_1|}.$$

Suppose we have $n_1, n_2, \ldots, n_{m-1}$ and $A_1, A_2, \ldots, A_{m-1}$ satisfying conditions of Lemma. Notice that α_{m-1} is of bounded type and that (3) is satisfied by only requiring that $n_m - n_{m-1}$ is sufficiently large. So by the exactly same procedure as above we choose a sufficiently large even number n_m and set

$$A_m = \max\{A; \text{for any } j \in \mathbb{Z} \text{ with } q_{n_m-1} \le |j| \le A q_{n_m-1}, \ \|D\hat{f}^j_{\alpha_m^A}\| < \theta_{|j|}\}.$$

□

Lemma 7. *Let $\beta_0, \beta_1, \beta_2 \in \mathbb{Q}/\mathbb{Z}$ be*

$$\beta_i = [b_1(\beta_i), b_2(\beta_i), \ldots, b_{2n}(\beta_i)] = \frac{p_{2n}(\beta_i)}{q_{2n}(\beta_i)}, \ i = 0, 1, 2$$

such that $\beta_0|[1, 2n-1] = \beta_1|[1, 2n-1] = \beta_2|[1, 2n-1]$ and for some $B \ge 1, B \in \mathbb{N}$, $b_{2n}(\beta_i) = B + i$.

Then for any $s_1, s_2 \in F^{-1}((\beta_0, \beta_2))$ and any $x \in S^1$ we have

$$\sum_{i=1}^{q_{2n}(\beta_2)} |(f_{s_1}^i(x), f_{s_2}^i(x))| \leq 7.$$

Proof. The argument of the proof is same as the Świątek's of lemma 3 in [Sw]. We recall Farey interval. A Farey interval is an interval $I = (\frac{p}{q}, \frac{p'}{q'}), p, p', q, q' \in \mathbb{Z}, q, q' > 0$ with $pq' - p'q = 1$. Then the following holds.

($*$) All rational in I have the form $\dfrac{kp + lp'}{kq + lq'}$, $k, l \geq 1, k, l \in \mathbb{N}$.

Since $q_{2n}(\beta_i) = (B+i)q_{2n-1}(\beta_0) + q_{2n-2}(\beta_0)$ and $p_{2n}(\beta_i) = (B+i)p_{2n-1}(\beta_0) + p_{2n-2}(\beta_0)$ two intervals $(\beta_0, \beta_1), (\beta_1, \beta_2)$ are Farey intervals and $q_{2n}(\beta_0) < q_{2n}(\beta_1) < q_{2n}(\beta_2)$ and by ($*$) the cardinality of the set of rationals in (β_0, β_2) with denominator less than $2q_{2n}(\beta_2)$ is at most six (three if $B \geq 3$).

For given $x \in S^1$ we define

$$t_1 = \sup\{t \in [\beta_{0-}, \beta_{0+}]; f_t^{q_{2n}(\beta_0)}(x) = x\},$$

$$t_2 = \inf\{t \in [\beta_{2-}, \beta_{2+}]; f_t^{q_{2n}(\beta_2)}(x) = x\}.$$

We define a diffeomorphism $G : S^1 \times [t_1, t_2] \to S^1 \times [t_1, t_2]$ by $G(y, t) = (f_t(y), t)$. Then we have

$$DG^i(y, t) = \begin{pmatrix} Df_t^i(y) & \frac{d}{dt}(f_t^i(y)) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Df_t^i(y) & 1 + \sum_{k=1}^{i-1} Df_t^{i-k}(f_t^k(y)) \\ 0 & 1 \end{pmatrix}.$$

So G monotonically twists S^1-direction to the right. More precisely, let $\tilde{G} : \mathbb{R} \times [t_1, t_2] \to \mathbb{R} \times [t_1, t_2], \tilde{G}(\tilde{y}, t) = (\tilde{f}_t(\tilde{y}), t)$ be a lift of G, then for any $i \geq 1$ the slope of the image of a vertical segment $\{\tilde{y}\} \times [t_1, t_2]$ by $\tilde{G}^i$ is everywhere positive finite. Let $P : S^1 \times [t_1, t_2] \to S^1$ be the projection on the first coordinate.

By contradiction we assume $\sum_{i=1}^{q_{2n}(\beta_2)} |(f_{s_1}^i(x), f_{s_2}^i(x))| > 7$. We consider the interval $\gamma = \{x\} \times [t_1, t_2]$ and its images by G^i. Since $[s_1, s_2] \subset (t_1, t_2)$, intervals $P(G^i(\gamma)), 1 \leq i \leq q_{2n}(\beta_2)$ overlap somewhere with multiplicity at least eight. Then, by the twist condition of G there exist distinct natural numbers i_k, $(0 \leq k \leq 7, k \in \mathbb{Z})$ with $1 \leq i_k \leq q_{2n}(\beta_2)$ such that for each k $(1 \leq k \leq 7)$,

$$(\{f_{t_2}^{i_0}(x)\} \times [t_1, t_2]) \cap G^{i_k}(\gamma) \neq \emptyset.$$

Moreover, using the preservation of order by $\tilde{f}_t : \mathbb{R} \times \{t\} \to \mathbb{R} \times \{t\}$ and the twist condition of G, we can see that for any $j \geq 0$,

$$(\{f_{t_2}^{i_0+j}(x)\} \times [t_1, t_2]) \cap G^{i_k+j}(\gamma) \neq \emptyset.$$

In particular for $j = q_{2n}(\beta_2) - i_0$ by the definition of t_2 we have

$$\gamma \cap G^{i_k + q_{2n}(\beta_2) - i_0}(\gamma) \neq \emptyset.$$

This imply that there exists a parameter value $u_k \in (t_1, t_2)$ such that $f_{u_k}^{q_{2n}(\beta_2)+i_k-i_0}(x) = x$. For each k $(1 \leq k \leq 7)$ the denominator of $\rho(f_{u_k})$ which divides $q_{2n}(\beta_2) + i_k - i_0$ is less than $2q_{2n}(\beta_2)$. This is a contradiction. $\square$

Proof of Theorem 2.
⋆ **Lower bound.** Let $j_m \in \mathbb{Z}$ be in Lemma 6 (2). Then $|j_m| \leq (A_m+1)q_{n_{m-1}} < q_{n_m}(\alpha_m^{A_m+2})$. We assume $j_m > 0$. Then since three rational numbers

$$\alpha_m^{A_m}|[1,n_m], \alpha_m^{A_m+1}|[1,n_m], \alpha_m^{A_m+2}|[1,n_m]$$

satisfy the condition of Lemma 7 and

$$\alpha_\infty \in (\alpha_m^{A_m}|[1,n_m], \alpha_m^{A_m+1}|[1,n_m]),$$

$$\alpha_m^{A_m+1} \in (\alpha_m^{A_m+1}|[1,n_m], \alpha_m^{A_m+2}|[1,n_m]),$$

we have for any $x \in S^1$

$$\begin{aligned}
&|\log D\hat{f}_{\alpha_\infty}^{j_m}(x) - \log D\hat{f}_{\alpha_m^{A_m+1}}^{j_m}(x)| \\
=&\left|\sum_{i=1}^{j_m-1} \log Df_0(\hat{f}_{\alpha_\infty}^i(x)) - \sum_{i=1}^{j_m-1} \log Df_0(\hat{f}_{\alpha_m^{A_m+1}}^i(x))\right| \\
\leq&\|D\log Df_0\| \sum_{i=1}^{j_m-1} |(\hat{f}_{\alpha_\infty}^i(x), \hat{f}_{\alpha_m^{A_m+1}}^i(x))| \leq 7\|D\log Df_0\|.
\end{aligned}$$

Since there exists $x_* \in S^1$ such that $|D\hat{f}_{\alpha_m^{A_m+1}}^{j_m}(x_*)| \geq \theta_{j_m}$ we have

$$\frac{\|D\hat{f}_{\alpha_\infty}^{j_m}\|}{\theta_{j_m}} \geq \frac{|D\hat{f}_{\alpha_\infty}^{j_m}(x_*)|}{|D\hat{f}_{\alpha_m^{A_m+1}}^{j_m}(x_*)|} \geq \exp(-7\|D\log Df_0\|).$$

For the case $j_m < 0$, using the chain rule $D\hat{f}_\alpha^{j_m}(x) = (D\hat{f}_\alpha^{-j_m}(\hat{f}_\alpha^{j_m}(x)))^{-1}$ we can obtain the same estimates .

As stated above by making the parameter a sufficiently large we can assume that $\|D\log Df_0\| = \|D\log Df_{a,0}\|$ is smaller than any given positive value.

⋆ **Upper bound.** Let $l \in \mathbb{Z}$ with $q_n \leq l < q_{n+1}$. The case $q_n \leq -l < q_{n+1}$ is similar. Let $n_m = \max\{n_i; n_i \leq n\}$. As in the proof of Theorem 1 we expand l as follows,

$$l = k_{n+1}q_n + \cdots + k_{n_m+1}q_{n_m} + cq_{n_m-1} + r,$$

where $0 \leq k_i \leq a_i(\alpha_\infty) = 1$ $(n_m+1 \leq i \leq n+1)$ and we choose $c \in \{-1,0,1\}$ so that $q_{n_m-1} \leq r \leq A_m q_{n_m-1}$.

By Lemma 2 (2) and Lemma 6 (1), (3) we have

$$\begin{aligned}
&\|D\hat{f}_{\alpha_\infty}^l\| \leq \|D\hat{f}_{\alpha_\infty}^{q_n}\|\cdots\|D\hat{f}_{\alpha_\infty}^{cq_{n_m-1}}\|\|D\hat{f}_{\alpha_\infty}^r\| \\
\leq&\exp(C\sum_{i=n_m-1}^{n}\lambda^i)(1+\|D\hat{f}_{\alpha_m}^r\|) \leq \exp(C\sum_{i=n_m-1}^{n}\lambda^i)(1+\theta_r).
\end{aligned}$$

Therefore we have

$$\limsup_{l\to\infty}\frac{\|D\hat{f}_{\alpha_\infty}^l\|}{\theta_l} \leq \limsup_{l\to\infty}\frac{\exp(C\sum_{i=n_m-1}^{n}\lambda^i)(1+\theta_r)}{\theta_l} \leq 1.$$

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